Numerical simulations of nonlinear wall-mode convection

Using Dedalus, a general framework for spectrally solving PDEs

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Introduction

Rotating Rayleigh-Benard convection, where a fluid body is heated from below and cooled from above while rotating around the vertical axis, is a quintessential model for studying rotationally-influenced convection.

The dynamics are controlled by the Rayleigh, Prandtl, and Taylor (or alternatively, Ekman) numbers:

$$\mathrm{Ra} \equiv \frac{g\alpha\Delta T H^3}{\nu\kappa} \qquad \mathrm{Pr} \equiv \frac{\nu}{\kappa} \qquad \mathrm{Ta} \equiv \frac{4\Omega^2 H^4}{\nu^2} \equiv \mathrm{E}^{-2}$$

The critical Rayleigh number for the onset of bulk convection, due to periodic modes in a horizontally unbounded domain, scales as (Chandrasekhar 1953):

$$\mathrm{Ra}_{\mathrm{c,bulk}} \sim 8.70 \times \mathrm{Ta}^{2/3} \quad \text{as} \quad \mathrm{Ta} \to \infty$$

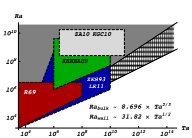
In a bounded domain, a different set of modes which are isolated to the container sidewalls (**wall modes**) can first trigger the onset of convection at high rotation rates (Goldstein 1993), with a critical Rayleigh number which scales as (Hermann 1993):

$$\mathrm{Ra}_{\mathrm{c,wall}} \sim 31.8 \times \mathrm{Ta}^{1/2} \quad \text{as} \quad \mathrm{Ta} \to \infty$$

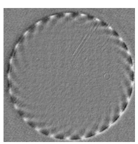
Julien (2007) demonstrate that the nonlinear dynamics of bulk convection can be well-captured via a dynamical reduction using multiple scales in the rapidly rotating limit. **Here we study an asymptotic reduction of the wall-mode dynamics under rapid rotation, where wall modes can be driven with**

$$\mathrm{Ra}/\mathrm{Ra}_{\mathrm{c,wall}} = \mathcal{O}(\mathrm{Ta}^{1/6})$$

while remaining subcritical to the onset of bulk convection.



Experimental studies of RRBC parameter space



Shadowgraph of wall modes in cylindrical container. Courtesy Bob Ecke.

Asymptotically Reduced Model

Using a multiple-scales analysis for a container with no-slip walls, fixed-temperature lids, and insulating sidewalls, the problem reduces to the following equations in the rapidly rotating limit:

- The **interior dynamics** are constrained by geostrophic and thermal wind balance:

$$g\alpha T = \partial_z P \qquad 2\Omega u = -\partial_y P \qquad 2\Omega v = \partial_x P$$

- The **baroclinic component** (temperature) evolves under two-dimensional (horizontal) advection and three-dimensional diffusion:

$$\partial_t T + u\partial_x T + v\partial_y T = \kappa(\partial_x^2 + \partial_y^2 + \partial_z^2)T$$

- The **barotropic component** is diagnostically determined by the balance of Ekman fluxes from the top and bottom lids:

$$(\partial_x^2 + \partial_y^2)\{P_{(z=0)} + P_{(z=Z)}\} = 0$$

- The **side layer dynamics** are reduced to nonlinear boundary conditions representing conservation of mass and heat between the interior flow and sidewall volume fluxes:

$$\left.\begin{aligned} \mp v &= \partial_x q_x + \partial_z q_z \\ \pm\kappa\partial_y T &= q_x\partial_x T + q_z\partial_z T \end{aligned}\right\} \quad y = 0, W$$

- The system is closed via a nonlocal constraint between the vertical and horizontal fluxes in the side layer, given by the Hilbert transform:

$$q_x = \mp H_z(q_z) \qquad y = 0, W$$

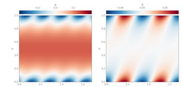
Here the Prandtl number is assumed to be a fixed number independent of the Taylor/Ekman number. The reduced dynamics are then controlled by a single parameter, the reduced Rayleigh number:

$$\mathrm{R} \equiv \mathrm{Ra}\,\mathrm{E} = \frac{g\alpha\Delta T H}{2\Omega\kappa} = \mathcal{O}(1)$$

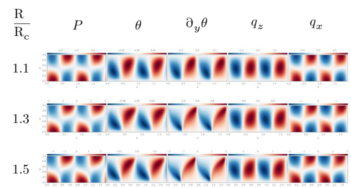
Simulation Results

We have performed preliminary simulations of the reduced model in a channel geometry for low-to-moderate supercriticalities, finding precessing saturated states:

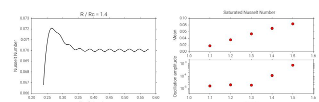
Typical planform: $\frac{\mathrm{R}}{\mathrm{R_c}} = 1.3$



Variables at sidewall (viewed from interior, traveling right):



The domain-integrated kinetic energy and sidewall Nusselt number, a non-dimensional measure of the sidewall convective heat flux relative to conduction in the interior, are found to oscillate in the saturated states, consistent with laboratory results:



Future work includes examining the long-time statistics for larger reduced Rayleigh numbers and cylindrical geometries.

Numerical Framework

(K. Burns, G. Vasil, J. Oishi, D. Lecoanet, B. Brown)

Our simulations were performed using **Dedalus**, a general framework for solving a broad range of PDEs using spectral methods, including initial-value, boundary-value, and generalized eigenvalue problems.

Dedalus uses **symbolic equation entry** to transform nearly arbitrary sets of equations, algebraic constraints, and boundary conditions into sparse and nearly banded matrix systems.

N-dimensional problems are supported, where at least the first (N-1) dimensions are represented by Fourier, sine, or cosine series, and the last dimension can be represented by Chebyshev polynomials. Generalized Jacobi polynomial and spherical harmonic bases are under development.

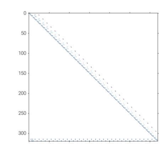
A broad range of mixed implicit-explicit timesteppers are available (and others are easily implemented), and non-constant coefficient linear terms are treated implicitly with spectral accuracy.

Dedalus is **MPI-parallelized** and written in Python 3, using compiled libraries for performance-critical routines.

Dedalus is **free and open-source software** with a friendly community and contributors from many different institutions. We are always glad to hear from people interested in using or contributing to the code. Learn more at:

http://dedalus-project.org/

One of the sparse matrix systems produced by parsing the asymptotically reduced model equations from plain text.



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